$SL_2(\mathbb{Z})$

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Contents

1 Definition

Definition 1.1 (Group Action) *A **group action** of* G *on a set* S *is a map* $G \times S \longrightarrow S$ *which satisfies the following axioms:*

a. *For any* $s \in S$, $e \cdot s = s$.

b. *For any* $g, g^{'} \in G$ *and* $s \in S$, $(gg') \cdot s = g \cdot (g' \cdot s)$.

Definition 1.2 (Orbit) *Let group* G *act on a set* S*. For any* $x \in X$*, the **orbit** of* x *is the set*

$$\mathrm{O}_x = \{y \in X \mid \exists\ g \in G\ s.t.\ g \cdot x = y\}.$$

Definition 1.3 (Stabilizer) *Let group* G *act on a set* S*. For any* $x \in X$*, the **stabilizer** of* x *is the subgroup of* G

$$\mathrm{G}_x = \{g \in G \mid g \cdot x = x\}.$$

2 Group Action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane

The **special linear group**,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}.$$

The **'upper half plane'**,

$$\bar{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\},\ \text{where } \mathcal{H} = \{x + iy \mid y > 0\}.$$

Now, we can define an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\bar{\mathcal{H}}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d},\ z \in \bar{\mathcal{H}}.$$

To check this is a group action,

a. If $\mathrm{Im}(z) > 0$, then $\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{|cz+d|^2}\,\mathrm{Im}(z) = \frac{1}{|cz+d|^2}\,\mathrm{Im}(z) > 0$.

b. $\mathrm{SL}_2(\mathbb{Z}) \times \bar{\mathcal{H}} \longrightarrow \bar{\mathcal{H}}$ is a group action.

   Let $z \in \bar{\mathcal{H}}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

   (i) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot z = z$

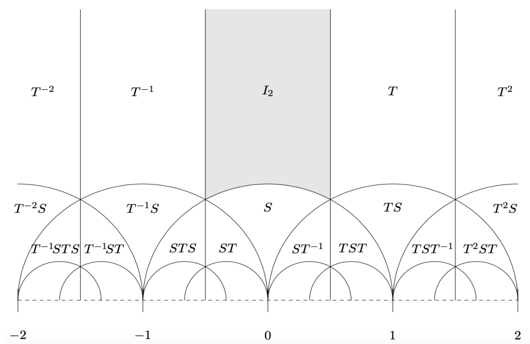
   (ii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\frac{ez+f}{gz+h}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot z \right)$

3 Generators of $\mathrm{SL}_2(\mathbb{Z})$

$$\mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The order of S is **4** and the order of T is **infinite**.

Theorem 3.1 $\mathrm{SL}_2(\mathbb{Z}) = \langle \mathrm{S}, \mathrm{ST} \mid \mathrm{S}^4 = (\mathrm{ST})^6 = \boldsymbol{e} \rangle$



Example 3.1 Express $\mathrm{A} = \begin{pmatrix} 8 & 11 \\ 5 & 7 \end{pmatrix}$ in terms of S and T.

4 Fundamental Domain

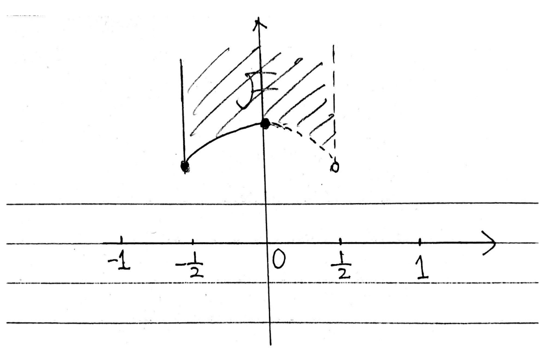
Suppose we have a group action of G on S.

Definition 4.1 *A* ***fundamental domain*** $\mathcal{F}$ *is a subset of* S *which contains* ***exactly*** *one point of each these orbits.*

Back to our example that the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\overline{\mathcal{H}}$, we claim

$$\mathcal{F} = \{\infty\} \cup \{z \in \bar{\mathcal{H}} \mid -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2} \text{ if } |z| > 1; \ \mathrm{Re}(z) \leq 0 \text{ if } |z| = 1\}$$

is a fundamental domain.



Q: How to prove $\mathcal{F}$ is a fundamental domain?

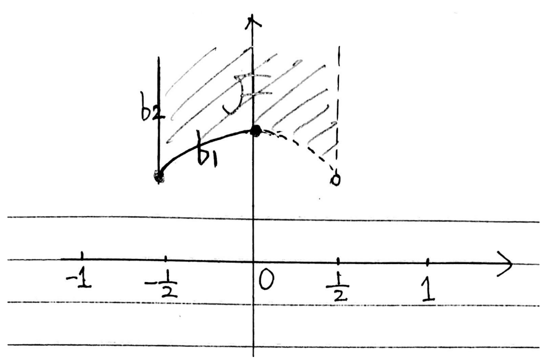
(a) Show that, for any $z \in \overline{\mathcal{H}}$, there exists a point $x \in \mathrm{O_z}$, such that $x \in \overline{\mathcal{H}}$.

(b) Show that, for any $z_1, z_2 \in \mathcal{F}$, $\mathrm{O_{z_1}} \neq \mathrm{O_{z_2}}$.

Let us sketch the proof of part (b).

Goal: to show that no such $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ s.t.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z_1 = z_2 \text{ where } z_1, z_2 \in \mathcal{F}$$



Case 1: $z_1, z_2 \in \mathcal{F} \setminus (b_1 \cup b_2)$
Case 2: $z_1 \in \mathcal{F},\ z_1 \in b_1$
Case 3: $z_1 \in \mathcal{F},\ z_1 \in b_2$
Case 4: $z_1 \in b_1,\ z_2 \in b_2$

5 Summary

Theorem 5.1 (Cool Topological Result)